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Approximating small and large amplitude periodic orbits of the oscillator $\ddot{x} + (1 + \dot{x}^2)x = 0$

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Abstract

It is shown that the oscillator $\ddot{x} + (1 + \dot{x}^2)x = 0$ is conservative and scaling laws for the period for small and large amplitude vibrations are derived. Analytical approximations of the periodic orbits are also constructed and these show excellent agreement with numerical solutions.

1. Introduction

Beatty and Mickens [1] and later Mickens [2] investigated the nonlinear oscillator

$$\ddot{x} + (1 + \dot{x}^2)x = 0 \tag{1}$$

subject to the initial conditions

$$x(0) = A > 0 \quad \text{and} \quad \dot{x}(0) = 0. \tag{2}$$

This oscillator is characterized by a velocity-dependent stiffness coefficient and depends on only one parameter. Mickens pointed out that while a periodic solution exists for all positive A and the period $T(A)$ approaches zero for large A [2], simple approximations for the period are only defined for a finite range of initial amplitudes. For example, the first-order harmonic balance approximation gives the following expression for the period:

$$T(A) = \sqrt{4 - A^2}, \tag{3}$$

which is valid only for $|A| < 2$. Even more advanced techniques, such as the harmonic balance base averaging of Chatterjee [3] give similar limitations. The properties of Eq. (1) significantly differ from those of the oscillator

$$\ddot{x} + (1 + \dot{x})x = 0, \tag{4}$$

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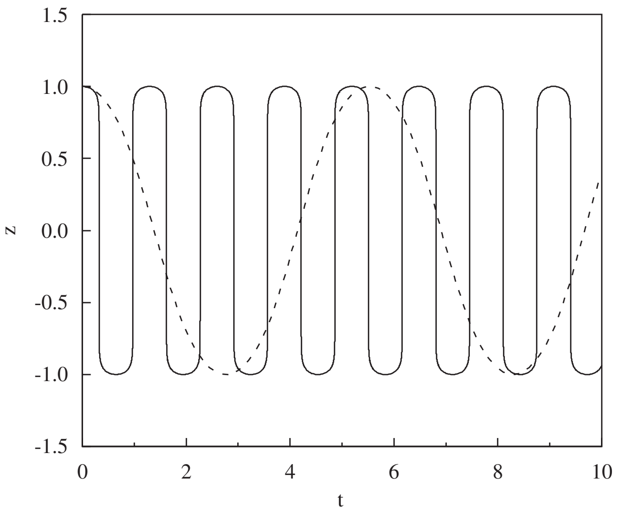

Fig. 1. The solution of Eqs. (1) and (2) is represented in terms of $z = x/A$ as a function of t. The oscillations look nearly harmonic for $A = 1$ (broken line) and square-wave-like for $A = 5$ (full line).

which has been studied in the context of relaxation oscillations [4] and laser oscillations [5]. For Eq. (4) subject to the initial conditions (2), the period of the oscillations increases monotonically with A and the output consists of short and intense pulses separated by intervals where x is almost zero. This is not the case of Eq. (1). As demonstrated in this note, the period of the oscillations for Eq. (1) decreases with A and the oscillations are reminiscent of square waves for large A (see Fig. 1).

Our objective is to determine approximations for the small and large A limit of the solution of Eqs. (1) and (2). Similar asymptotic representations of the period have recently been published in Ref. [6]. Here, it is first shown that this oscillator is conservative and depends on a single-well potential. The period of oscillation is then expressed in an integral form and the leading terms in its expansion for small and large A are determined. Analytical constructions of small and large amplitude periodic solutions directly from Eq. (1) are also provided.

2. Calculating the period

As the first step of the analysis, Eq. (1) will be nondimensionalized. The usefulness of this procedure is discussed by Mickens [7]. The significance of the rescaling in this case is that parameter A will appear in the differential equation rather than the initial conditions. This is utilized in our proof of conservativeness and in the construction of the potential for the system.

With the scaled variable $z = x/A$ Eqs. (1) and (2) are rewritten as

$$\ddot{z} + (1 + A^2\dot{z}^2)z = 0 \quad \text{with } z(0) = 1, \quad \dot{z}(0) = 0, \tag{5}$$

or equivalently, the following system of first-order differential equations:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -(1 + A^2y^2)z, \tag{6}$$

$$z(0) = 1, \quad y(0) = 0. \tag{7}$$

By dividing the two equations of Eq. (6) ($z \neq 0$), we obtain an equation for the trajectory $y = y(z)$ satisfying

$$\frac{\mathrm{d}y}{\mathrm{d}z} = -\frac{(1 + A^2y^2)z}{y}, \quad y(1) = 0. \tag{8}$$

This equation is separable and admits the first integral

$$\ln(1+A^2y^2)+A^2z^2=A^2. \tag{9}$$

From Eq. (9) $1+A^2y^2$ is expressed as

$$1+A^2y^2=\exp(A^2(1-z^2)). \tag{10}$$

Differentiating both sides and simplifying ($y=\dot{z}$) leads to the following second-order differential equation for z:

$$\ddot{z}+\mathrm{e}^{A^2(1-z^2)}z=0. \tag{11}$$

It is easy to show that this equation is equivalent to

$$\ddot{z}+\frac{\mathrm{d}V}{\mathrm{d}z}=0, \tag{12}$$

where $V(z)$ is a single-well potential defined by ($V(1)=0$)

$$V(z)\equiv\frac{1-\exp(A^2(1-z^2))}{2A^2}. \tag{13}$$

The existence of the potential $V(z)$ provides a simple proof of the conservativeness [8] of the equivalent oscillators (11) and (1). Using Eq. (10), we write

$$y=\dot{z}=\pm A^{-1}\sqrt{\exp(A^2(1-z^2))-1}. \tag{14}$$

This allows to define the period as [2]

$$T(A)=4A\int_0^1\frac{\mathrm{d}z}{\sqrt{\exp(A^2(1-z^2))-1}}. \tag{15}$$

The small A limit is obtained by expanding the exponential in Eq. (15) for small A^2. Integrating the first two terms gives

$$T(A)\simeq 2\pi-\frac{A^2\pi}{4}=2\pi\left(1-\frac{A^2}{8}\right). \tag{16}$$

The large A limit is more delicate to compute, but the main contribution of the integral comes for z close to 1. Introducing the new variable u defined by

$$z=1+A^{-2}u, \tag{17}$$

Eq. (15) becomes

$$T(A)=\frac{4}{A}\int_{-A^2}^{0}\frac{\mathrm{d}u}{\sqrt{\mathrm{e}^{-2u-u^2/A^2}-1}}. \tag{18}$$

In the $A\to\infty$ limit this integral has a closed form solution [9]

$$\lim_{A\to\infty}\int_{-A^2}^{0}\frac{\mathrm{d}u}{\sqrt{\mathrm{e}^{-2u-u^2/A^2}-1}}=\frac{\pi}{2}. \tag{19}$$

Therefore the leading asymptotic behavior of the period for large A is

$$T(A)\simeq\frac{2\pi}{A}. \tag{20}$$

Fig. 2 compares the exact numerical period with the two approximations by Eqs. (16) and (20).

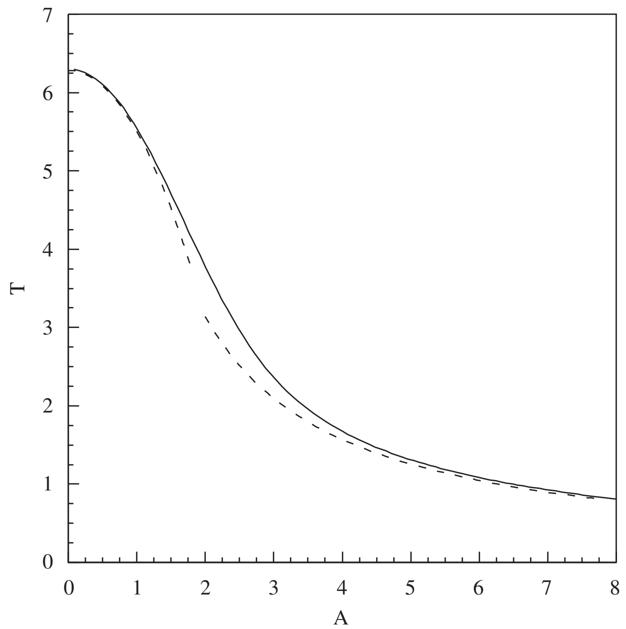

Fig. 2. The numerically computed period $T(A)$ (full line) is compared to its small A and large A asymptotic approximations (broken lines) given by Eqs. (16) and 20, respectively.

3. Approximations of periodic orbits

The conservative nature of the oscillator Eq. (11) is now exploited to construct analytical approximations of the periodic solutions for small and large A.

For $A \ll 1$

$$e^{A^2(1-z^2)} \approx 1 + A^2 - A^2 z^2 \tag{21}$$

and thus Eq. (11) can be written as

$$\ddot{z} + (1 + A^2)z - A^2 z^3 = 0. \tag{22}$$

Closed form solutions of this equation can be found in terms of Jacobi elliptic functions [10]. Substituting a solution of the form (a_1 , a_2 are nonzero)

$$z(t) = a_1 \operatorname{sn}(u, m), \quad u = a_2 t + b, \tag{23}$$

into Eq. (22) yields

$$-1 - A^2 + a_2^2(m \operatorname{cn}^2(u, m) + \operatorname{dn}^2(u, m)) + A^2 a_1^2 \operatorname{sn}^2(u, m) = 0. \tag{24}$$

Using the identities

$$\operatorname{sn}^2(u, m) + \operatorname{cn}^2(u, m) = 1, \tag{25}$$

$$m \operatorname{sn}^2(u, m) + \operatorname{dn}^2(u, m) = 1 \tag{26}$$

leads to

$$a_2^2(1 + m) - 1 - A^2 + (A^2 a_1^2 - 2a_2^2 m) \operatorname{sn}^2(u, m) = 0. \tag{27}$$

Equating the coefficients of this equation with zero provides two algebraic equations:

$$a_2^2(1 + m) - 1 - A^2 = 0, \tag{28}$$

$$A^2 a_1^2 - 2a_2^2 m = 0. \tag{29}$$

Two additional equations are provided by the initial conditions (5) and therefore the four unknowns a_1, a_2, m, b can be solved for as

$$\begin{aligned} a_1 &= 1, \\ a_2 &= \sqrt{1 + A^2/2}, \\ m &= A^2/(2 + A^2), \\ b &= \mathrm{sn}^{-1}(1, m). \end{aligned} \tag{30}$$

Fig. 3 shows the periodic solution of Eq. (11) for $A = 0.5$ and the corresponding approximation by Eqs. (23) and (30).

The analysis of the period for large A indicates that the solution is mostly in the neighborhood the slow manifolds near $z = \pm 1$ with the time scale $t = O(A^{-1})$. This motivates the introduction of the new time $s = At$ and $z = \pm 1 + A^{-2}u$ into Eq. (6). The A^2y^2 term in the equation for y also suggests the change of variable $y = A^{-1}v$. For large A, the equation for y then reduces to

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \mp(1 + v^2), \quad v(0) = 0. \tag{31}$$

Eq. (31) admits the solution

$$v = \mp \tan(s). \tag{32}$$

This expression becomes unbounded as $s \to \pm\pi/2$, where the fast jump transitions between $z = \pm 1$ occur. As a consequence, the half-period in s is the time interval between two successive jumps and equals π. Equivalently, the half-period in t equals π/A, as seen from Eq. (20). The following initial value problem for u is then obtained from the first equation of Eq. (6)

$$\mathrm{d}u/\mathrm{d}s = v, \quad u(0) = 0. \tag{33}$$

The solution is

$$u = \mp \ln|\sec(s)|. \tag{34}$$

For $A = 5$, the approximation of the slow manifolds near $z = \pm 1$ are given by

$$z = -1 + A^{-2}\ln|\sec(2\pi t/T_0)|, \quad T_0/4 < t < 3T_0/4, \tag{35}$$

$$z = 1 - A^{-2}\ln|\sec(2\pi t/T_0)|, \quad 3T_0/4 < t < 5T_0/4, \tag{36}$$

where $T_0 = 2\pi/A \simeq 1.3$. Fig. 4 shows excellent agreement between the numerical solution and the approximation.

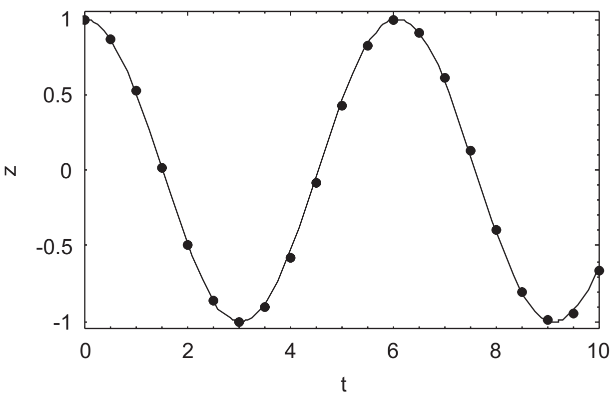

Fig. 3. The periodic solution of Eq. (11) for $A = 0.5$ (full line) is compared with its asymptotic approximation (dots) given by Eqs. (23) and (30).

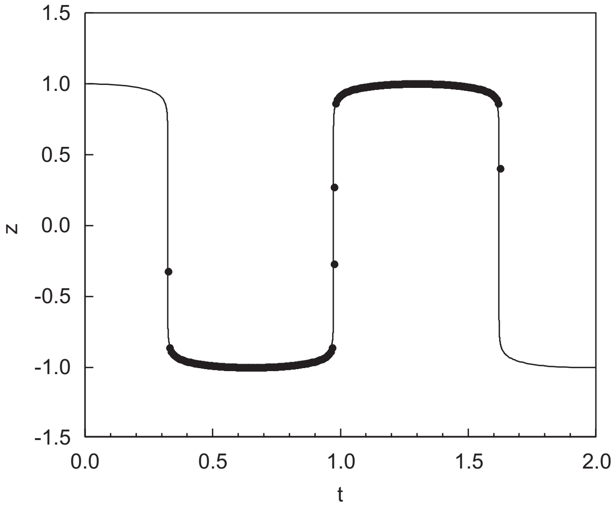

Fig. 4. The periodic solution of Eq. (11) for $A = 5$ (full line) is compared with its asymptotic approximation (dots) given by Eqs. (35) and (36).

4. Discussion

In summary, our analysis of Eqs. (1) and (2) demonstrated that these equations describe a conservative oscillator. For small amplitude oscillations the period quadratically depends on the vibration amplitude, i.e. $T(A) \simeq 2\pi(1 - A^2/8)$. Further, a good analytical approximation of the motion is also given in terms of the Jacobi elliptic function. For large A the period scales like A^{-1}, and the oscillations are square-wave-like, switching periodically between the slow manifolds $x \sim \pm A$. Numerical solutions for $A = 0.5$ and $A = 5$ compare quantitatively well with the analytical approximations.

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